2.3: Applications

A *mathematical model* is an equation or set of equations that mimic some phenomenon.

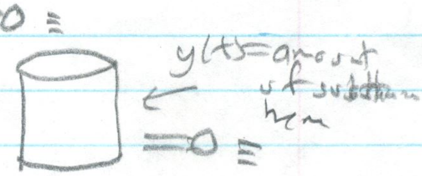
Disclaimer: Not claiming perfect fit.

For this course
1. Know applications from book, homework and lecture.
2. Be able to translate basic statements ("rate", "proportional to")

CASE STUDIES

A mixing problem $\qquad \frac{dy}{dt} = \text{Rate IN} - \text{Rate out}$

Ex) A 1000 L tank initially contains 400 L of water, in which 50 kg of salt is desolved.



IN: Fluid is coming in at 30 L/min and it contains 0.3 kg/L of salt (30% salt)

OUT: Tank drains at 10 L/min.

How much salt is in the tank when the level reaches the top?

SOL'N

STEP 1 Volume? $\quad V(t) = 400 + 20t \quad$ (why?)

STEP 2 Rate IN $= 0.3\,\frac{kg}{L} \cdot 30\,L/min = 9\,kg/min$

$$\text{Rate OUT} = \underbrace{\frac{y}{V(t)}}_{????}\,10\,L/min = \frac{y}{410+20t}\,10 = \frac{y}{40+2t}$$

$$\frac{dy}{dt} = 9 - \frac{y}{40+2t} \qquad y(0) = 50$$

SOLVE!

[1] $\frac{dy}{dt} + \frac{1}{40+2t} y = 9 \qquad p(t) = \frac{1}{40+2t}$

[2] $\mu(t) = e^{\int \frac{1}{40+2t} dt} = e^{\frac{1}{2}\ln|40+2t|} = \sqrt{40+2t}$

[3] $\sqrt{40+2t}\,\frac{dy}{dt} + \frac{1}{\sqrt{40+2t}}\, y = 9\sqrt{40+2t}$

$$\frac{d}{dt}\left(\sqrt{40+2t}\; y\right) = 9\sqrt{40+2t}$$

[4] $\sqrt{40+2t}\; y = 9\int \sqrt{40+2t}\, dt$ $\qquad u = 40+2t,\ du = 2dt,\ dt = \frac{1}{2}du$

$$= \frac{9}{2}\int u^{1/2} du$$

$$= \frac{9}{2}\frac{2}{3} u^{3/2} + c$$

$$\sqrt{40+2t}\; y = 3(40+2t)^{3/2} + c$$

$$\boxed{y = 3(40+2t) + \frac{c}{\sqrt{40+2t}}}$$

[5] $y(0) = 50 \Rightarrow 50 = 3(40) + \frac{c}{\sqrt{40}}$

$$-70 = \frac{c}{\sqrt{40}} \qquad c = -70\sqrt{40}$$

$$\boxed{y = 120 + 6t - \frac{70\sqrt{40}}{\sqrt{40+2t}}}$$

TANK FULL $\Rightarrow 400 + 20t = V(t) = 1000$

$$\Rightarrow 20t = 600 \Rightarrow t = 30 \text{ min}$$

$$y(30) = 120 + 6\cdot 30 - \frac{70\sqrt{40}}{\sqrt{40+2(30)}} \approx \boxed{135.726 \text{ kg}}$$

STOP SPRING 2015

BOARD
- HW1 IN PILE AT FRONT
- PICK UP TP1 & HW2 IF YOU DIDN'T LAST TIME

MATH 307

2.3: More Application Examples

- Temp
- Interest bearing
- Air resistance

Temperature $\frac{dT}{dt} = \underbrace{k}_{\text{proportional to}} \underbrace{(T - T_s)}_{\text{temperature difference between object and surroundings}}$

constant temp

$$\frac{dT}{dt} = -0.1 (T - 70)$$

$$T(0) = 200$$

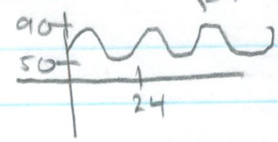
separate or int. factor ⇓

$$T(t) = 70 + 130 e^{-0.1t}$$

↓

SHOW PICTURE

$$T_s(t) = 70 + 20\sin\left(\frac{2\pi}{24} t\right) = \text{temp outside}$$



$$\frac{dT}{dt} = -0.1 (T - T_s(t))$$

$$\frac{dT}{dt} = -0.1 \left(T - 70 - 20\sin\left(\frac{2\pi}{24} t\right)\right)$$

$$T(0) = 200$$

$$T(t) \approx 137 e^{-0.1t} + 2.5\sin\left(\frac{2\pi}{24} t\right) - 7\cos\left(\frac{2\pi}{24} t\right) + 70$$

↓

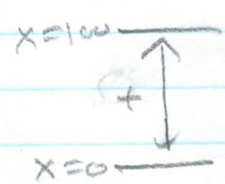
SHOW PICTURE

DO LAST →

DO FINANCE NEXT

AIR RESISTANCE

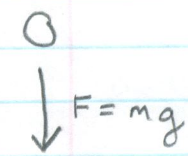
$$ma = F \Rightarrow m\frac{dv}{dt} = F$$



NO AIR RESISTANCE

IF we define up as increasing x (positive velocity)

THEN $m\frac{dv}{dt} = -mg$ $\quad g = 9.8\ m/s^2 = 32\ ft/s^2$

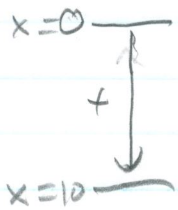


↳ $v = -gt + v_0$

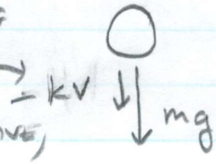
↳ $x = -\frac{g}{2}t^2 + v_0 t + h_0$

Air resistence — A common model:

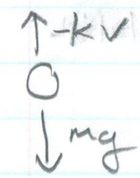
Force due to air resistance is proportional to velocity. In opposite in direction to velocity



Object moving upward: "drag coeff" →



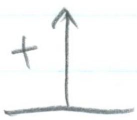
Object moving downward



If we define down as positive,

$$m\frac{dv}{dt} = +mg - kv$$

Note v negative ⇒ both positive ← common in textbooks



If we define up as positive: Note v positive ⇒ got negative

Opposite velocity in either case

$$m\frac{dv}{dt} = -mg - kv$$

$$\boxed{m\frac{dv}{dt} = -mg - kv \Rightarrow \frac{dv}{dt} + \frac{k}{m}v = -g}$$

$$\mu(t) = e^{\int \frac{k}{m}dt} = e^{k/m\, t}$$

$$\frac{d}{dt}\left(e^{k/m\, t} v\right) = \int -g e^{k/m\, t} dt$$

$$e^{k/m\, t} v = -\frac{mg}{k} e^{k/m\, t} + C$$

$$\boxed{v = -\frac{mg}{k} + Ce^{-\frac{k}{m}t}}$$

$v(0) = v_0 \Rightarrow v_0 = \frac{-mg}{k} + C \qquad C = v_0 + mg/k$

$$\boxed{v(t) = \frac{-mg}{k} + (v_0 + mg/k)e^{-\frac{k}{m}t}}$$

$$h(t) = \int \frac{-mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-k/m\, t}\, dt$$

$$\boxed{h(t) = \frac{-mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-k/m\, t} + C}$$

↳ Example with #'s → terminal velocity

N8DE

Ex] An object (spherical) of mass $m = 1$ kg is shot straight up in the air from the ground with an initial velocity of 250 m/sec (≈ 559 mph). Assume air resistance is modeled as above. with $k = 0.2$ kg/s
(Note $g = 9.8$ m/sec^2)

↑ reasonable for a sphere in air

What is the maximum height?

$m = 1,\ g = 9.8,\ v_0 = 250,\ k = 0.2$

$$m\frac{dv}{dt} = -mg - kv \Rightarrow \boxed{\frac{dv}{dt} = -9.8 - 0.2v}$$

$$v(t) = \frac{-mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-k/m\,t} \Rightarrow v(t) = \frac{-9.8}{0.2} + \left(250 + \frac{9.8}{0.2}\right)e^{-\frac{0.2}{1}t}$$

$$\boxed{v(t) = -49 + 299e^{-0.2t}}$$

NOTE $v(0) = 250$!

$$h(t) = -49t - 1495e^{-0.2t} + C$$

$$h(0) = 0 \Rightarrow 0 = 0 - 1495 + C \qquad C = 1495$$

$$\boxed{h(t) = -49t - 1495e^{-0.2t} + 1495}$$

MAX HEIGHT OCCURS WHEN $h'(t) \overset{?}{=} 0 \Leftrightarrow v(t) = 0$

$$-49 + 299e^{-0.2t} \overset{?}{=} 0$$

$$e^{-0.2t} = \frac{49}{299}$$

$$-0.2t = \ln(49/299)$$

$$t = \frac{-1}{0.2}\ln(49/299) \approx 9.04312 \text{ sec}$$

$$h(9.043116) \approx \boxed{1806.89 \text{ m}}$$

Aside: THE NO AIR RESISTANCE MODEL

- $\frac{dv}{dt} = -9.8 \Rightarrow v_1(t) = -9.8t + 250$

$$h(t) = -4.9t^2 + 250t + 0$$

predicts max height $= h\left(\frac{250}{9.8}\right) \approx \boxed{3188.78 \text{ m}}$ ft

Terminal velocity: NOTE as $\lim_{t\to\infty} v(t) = \underbrace{-\frac{mg}{k}}$ CALLED TERMINAL VELOCITY.

$\boxed{\text{OTHER APPS}}$

SAVINGS AND LOANS

CONSIDER AN ACCOUNT THAT HAS A BALANCE THAT IS CHANGING IN TWO WAYS

1. DEPOSITS/PAYMENTS MADE THROUGHOUT THE YEAR THAT TOTAL $\pm K$ dollars/year
2. COMPOUND INTEREST WITH A DECIMAL RATE OF r ANNUALLY (COMPOUNDED CONTINUOUSLY) ADDS ry dollars/year in interest.

THEN $\boxed{\frac{dy}{dt} = ry \pm k}$

ADDING TO BALANCE.

PAYING DOWN BALANCE

Ex) You are paying back \$30,000 in student loans. The interest rate is 5% annually, compounded continuously. HOW MUCH SHOULD YOU PAY EACH YEAR IN ORDER TO PAY OFF THE LOAN IN 10 years?

$\frac{dy}{dt} = 0.05y - k$ $\qquad y(0) = 30000 \qquad y(10) = 0.$

$k = ???$

$y(t) = 20k + Ce^{0.05t}$

ASIDE $0.05 \cdot 30000 = \$1500$ interest in first year

$30000 = 20k + Ce^{0} \Rightarrow C = 30000 - 20k$

$y(t) = 20k + (30000 - 20k)e^{0.05t}$

$0 = 20k + (30000 - 20k)e^{0.5}$

$0 = 20k + 30000e^{0.5} - 20ke^{0.5}$

$20ke^{0.5} - 20k = 30000e^{0.5}$

$k(20e^{0.5} - 20) = 30000e^{0.5}$

$\boxed{k = \frac{30000e^{0.5}}{20e^{0.5} - 20} \approx 3812.24}$

ALMOST EXACTLY WHAT A LOAN PAYOFF CALCULATOR GIVES

\$3812.24 / year ($\approx$ \$317.69 / month)

$$\text{TOTAL PAYMENTS} = \frac{\$3812.24}{\text{yr}} \cdot 10 \text{ years} = \boxed{\$38122.40}$$

$$\underbrace{38122.40 - 30000}_{30000 = \text{original debt}} = \boxed{8{,}122.40} \text{ IN INTEREST}$$

HW QUESTIONS?